

一. 填空题

1. 设 $x_n = \begin{cases} \frac{n}{n+1}, & n \text{ 为奇数} \\ \frac{n+1}{n}, & n \text{ 为偶数} \end{cases}$, 则极限 $\lim_{n \rightarrow \infty} x_n =$ _____

解: 由 $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$, $\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$ 得 $\lim_{n \rightarrow \infty} x_n = 1$

2. 极限 $\lim_{n \rightarrow \infty} \frac{1+2+\dots+n}{n^2} =$ _____

解: $\lim_{n \rightarrow \infty} \frac{1+2+\dots+n}{n^2} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2}n(n+1)}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{n+1}{n} = \frac{1}{2}$

3. 极限 $\lim_{x \rightarrow 0} \frac{(3x+1)^3(x+3)^2}{(2x+5)^5} =$ _____

解: $\lim_{x \rightarrow 0} \frac{(3x+1)^3(x+3)^2}{(2x+5)^5} = \lim_{x \rightarrow 0} \frac{(3+\frac{1}{x})^3(1+\frac{2}{x})^2}{(2+\frac{5}{x})^5} = \frac{3^3 \cdot 1^2}{2^5} = \frac{27}{32}$

4. 极限 $\lim_{x \rightarrow +\infty} (\sqrt{4x^2+3x} - 2x) =$ _____

解: $\lim_{x \rightarrow +\infty} (\sqrt{4x^2+3x} - 2x)$

$= \lim_{x \rightarrow +\infty} \frac{(\sqrt{4x^2+3x} - 2x)(\sqrt{4x^2+3x} + 2x)}{\sqrt{4x^2+3x} + 2x}$

$= \lim_{x \rightarrow +\infty} \frac{3x}{\sqrt{4x^2+3x} + 2x}$

$= \lim_{x \rightarrow +\infty} \frac{3}{\sqrt{4+\frac{3}{x}} + 2} = \frac{3}{4}$

同步练习 P₁₀

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二. 计算题

1. 求极限 $\lim_{x \rightarrow 3} (\frac{1}{x-3} - \frac{6}{x^2-9})$

解: $\lim_{x \rightarrow 3} (\frac{1}{x-3} - \frac{6}{x^2-9}) = \lim_{x \rightarrow 3} \frac{(x+3) - 6}{x^2-9} = \lim_{x \rightarrow 3} \frac{1}{x+3} = \frac{1}{6}$

2. 求极限 $\lim_{x \rightarrow 0^+} \frac{1-e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}$

解: $\lim_{x \rightarrow 0^+} \frac{1-e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} = \lim_{x \rightarrow 0^+} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{e^{\frac{1}{x}}} - 1}{\frac{1}{e^{\frac{1}{x}}} + 1} = \frac{0-1}{0+1} = -1$

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3. 求极限 $\lim_{n \rightarrow \infty} [\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \dots + \frac{1}{(2n-1) \times (2n+1)}]$

解: $\because \frac{1}{(2k-1)(2k+1)} = \frac{1}{2} (\frac{1}{2k-1} - \frac{1}{2k+1})$

$\therefore \lim_{n \rightarrow \infty} [\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \dots + \frac{1}{(2n-1)(2n+1)}]$

$= \lim_{n \rightarrow \infty} [\frac{1}{2}(\frac{1}{1} - \frac{1}{3}) + \frac{1}{2}(\frac{1}{3} - \frac{1}{5}) + \dots + \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1})]$

$= \lim_{n \rightarrow \infty} \frac{1}{2} (1 - \frac{1}{2n+1})$

$= \frac{1}{2} (1 - 0)$

$= \frac{1}{2}$

4. 求极限 $\lim_{n \rightarrow \infty} [(1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \cdots (1 - \frac{1}{n^2})]$

$$\text{解: } \because 1 - \frac{1}{k^2} = (1 - \frac{1}{k})(1 + \frac{1}{k}) = \frac{k-1}{k} \cdot \frac{k+1}{k}$$

$$\therefore \lim_{n \rightarrow \infty} [(1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \cdots (1 - \frac{1}{n^2})]$$

$$= \lim_{n \rightarrow \infty} [(\frac{1}{2} \cdot \frac{3}{2}) \cdot (\frac{2}{3} \cdot \frac{4}{3}) \cdot (\frac{3}{4} \cdot \frac{5}{4}) \cdots (\frac{n-1}{n} \cdot \frac{n+1}{n})]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{n+1}{n} = \frac{1}{2}$$

5. 求极限 $\lim_{x \rightarrow -\infty} (\sqrt{x^2+x-1} - \sqrt{x^2-2x+3})$

$$\text{解: } \lim_{x \rightarrow -\infty} (\sqrt{x^2+x-1} - \sqrt{x^2-2x+3})$$

$$= \lim_{x \rightarrow -\infty} \frac{(\sqrt{x^2+x-1} - \sqrt{x^2-2x+3})(\sqrt{x^2+x-1} + \sqrt{x^2-2x+3})}{(\sqrt{x^2+x-1} + \sqrt{x^2-2x+3})}$$

$$= \lim_{x \rightarrow -\infty} \frac{3x - 4}{\sqrt{x^2+x-1} + \sqrt{x^2-2x+3}}$$

$$= \lim_{x \rightarrow -\infty} \frac{-3 + \frac{4}{x}}{\sqrt{1 + \frac{1}{x} - \frac{1}{x^2}} + \sqrt{1 - \frac{2}{x} + \frac{3}{x^2}}}$$

(分子分母同除 $\sqrt{x^2} = |x| = -x$
这里搞错, 答案就是 $\frac{3}{2}$ 了。

$$= \frac{-3 + 0}{1 + 1} = -\frac{3}{2}$$

同步练习 P12

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6. 求极限 $\lim_{n \rightarrow \infty} \frac{3^n + 4^n}{3^{n+1} + 4^{n+1}}$

$$\begin{aligned} \text{解: } \lim_{n \rightarrow \infty} \frac{3^n + 4^n}{3^{n+1} + 4^{n+1}} &= \lim_{n \rightarrow \infty} \frac{\left(\frac{3}{4}\right)^n \cdot \frac{1}{4} + \frac{1}{4}}{\left(\frac{3}{4}\right)^n + 1} \quad (\text{分子分母同除以 } 4^{n+1}) \\ &= \frac{0 + \frac{1}{4}}{0 + 1} = \frac{1}{4} \end{aligned}$$

7. 求极限 $\lim_{n \rightarrow \infty} [(1+a)(1+a^2)(1+a^4) \cdots (1+a^{2^n})]$ (其中 $|a| < 1$)

$$\text{解: } \lim_{n \rightarrow \infty} (1+a)(1+a^2)(1+a^4) \cdots (1+a^{2^n})$$

$$= \lim_{n \rightarrow \infty} \frac{(1-a)(1+a)(1+a^2)(1+a^4) \cdots (1+a^{2^n})}{(1-a)}$$

$$= \lim_{n \rightarrow \infty} \frac{(1-a^2)(1+a^2)(1+a^4) \cdots (1+a^{2^n})}{1-a}$$

$$= \lim_{n \rightarrow \infty} \frac{(1-a^4)(1+a^4) \cdots (1+a^{2^n})}{1-a}$$

$$= \lim_{n \rightarrow \infty} \frac{1-a^{2^{n+1}}}{1-a}$$

$$= \frac{1-0}{1-a} = 1-a$$

同步练习 P₁₂

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8. 已知 $\lim_{x \rightarrow \infty} \left(\frac{2x^2+3x-1}{x+1} - ax - b \right) = 2$, 求 a, b .

解: 方法一,

$$\begin{aligned} \text{由 } \lim_{x \rightarrow \infty} \left(\frac{2x^2+3x-1}{x+1} - ax - b \right) &= \lim_{x \rightarrow \infty} \frac{2x^2+3x-1 - (ax+b)(x+1)}{x+1} \\ &= \lim_{x \rightarrow \infty} \frac{2x^2+3x-1 - ax^2 - ax - bx - b}{x+1} = \lim_{x \rightarrow \infty} \frac{(2-a)x^2 + (3-a-b)x - 1-b}{x+1} = 2 \end{aligned}$$

$$\Rightarrow \begin{cases} 2-a=0 \\ 3-a-b=2 \end{cases} \Rightarrow \begin{cases} a=2 \\ b=-1 \end{cases}$$

方法二

$$\text{由 } \lim_{x \rightarrow \infty} \left(\frac{2x^2+3x-1}{x+1} - ax - b \right) = 2 \text{ 得 } \lim_{x \rightarrow \infty} \left[\frac{2x^2+3x-1}{x+1} - (ax+b+2) \right] = 0$$

即 $y = ax + b + 2$ 是曲线 $y = \frac{2x^2+3x-1}{x+1}$ 的渐近线

$$\text{所以 } a = \lim_{x \rightarrow \infty} \left[\frac{2x^2+3x-1}{x+1} / x \right] = 2$$

$$b+2 = \lim_{x \rightarrow \infty} \left[\frac{2x^2+3x-1}{x+1} - 2x \right] = 1 \Rightarrow b = -1$$

$$\text{故 } \begin{cases} a=2 \\ b=-1 \end{cases}$$

同步练习 P13

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9. 设 $f(x) = \begin{cases} \frac{x^2+ax+b}{x-1}, & x > 1 \\ 2x+1, & x \leq 1 \end{cases}$ 且 $\lim_{x \rightarrow 1} f(x)$ 存在, 求 a, b

解: $\because \lim_{x \rightarrow 1} f(x)$ 存在 $\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (2x+1) = 3$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{x^2+ax+b}{x-1} = \lim_{x \rightarrow 1^+} f(x) = 3 \Rightarrow$$

$$\lim_{x \rightarrow 1^+} (x^2+ax+b) = 1+a+b = 0, \text{ 则}$$

$$\lim_{x \rightarrow 1^+} \frac{x^2+ax+b}{x-1} = \lim_{x \rightarrow 1^+} \frac{x^2-(b+1)x+b}{x-1} = \lim_{x \rightarrow 1^+} (x-b) = 1-b = 3$$

$$\therefore \begin{cases} 1+a+b = 0 \\ 1-b = 3 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = -2 \end{cases}$$

10. 求极限 $\lim_{n \rightarrow \infty} [\sqrt{1+2+\dots+n} - \sqrt{1+2+\dots+(n-1)}]$

解: 原式 = $\lim_{n \rightarrow \infty} \left[\sqrt{\frac{n(n+1)}{2}} - \sqrt{\frac{n(n-1)}{2}} \right]$

$$= \lim_{n \rightarrow \infty} \frac{(\sqrt{n(n+1)} + \sqrt{n(n-1)})}{(\sqrt{n(n+1)} - \sqrt{n(n-1)}) (\sqrt{n(n+1)} + \sqrt{n(n-1)})} \cdot \frac{1}{\sqrt{2}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sqrt{n^2+n} + \sqrt{n^2-n}}{2n} \cdot \frac{1}{\sqrt{2}} = \lim_{n \rightarrow \infty} \frac{\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n}}}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}$$

同步练习 P13

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11. 求极限 $\lim_{n \rightarrow \infty} \left[\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} \right]$

解: $\because \frac{k-1}{k!} = \frac{k}{k!} - \frac{1}{k!} = \frac{1}{(k-1)!} - \frac{1}{k!}$ 这里 $k \geq 2$

$$\therefore \lim_{n \rightarrow \infty} \left[\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\left(\frac{1}{1!} - \frac{1}{2!} \right) + \left(\frac{1}{2!} - \frac{1}{3!} \right) + \left(\frac{1}{3!} - \frac{1}{4!} \right) + \cdots + \left(\frac{1}{n!} - \frac{1}{(n+1)!} \right) \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{1}{1!} - \frac{1}{(n+1)!} \right] = 1 - 0 = 1$$

同步练习 P14 三. 证明题

1. 设 $\lim_{x \rightarrow \infty} f(x)$ 存在, 证明: 存在 $\delta > 0$, $M > 0$, 当 $|x| > \delta$ 时, 有 $|f(x)| \leq M$

证明: 不妨设 $\lim_{x \rightarrow \infty} f(x) = A$, 则由极限的定义知, $\exists \delta > 0$,

当 $|x| > \delta$ 时, $|f(x) - A| < 1$, 即 $-1 + A < f(x) < 1 + A$

取 $M = \max \{ |-1 + A|, |1 + A| \}$, 则当 $|x| > \delta$ 时, 有 $|f(x)| \leq M$

同步练习 P₁₄ 三. 2

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2. 设 $\lim_{x \rightarrow x_0} f(x)$, $\lim_{x \rightarrow x_0} g(x)$ 存在, 证明: $\lim_{x \rightarrow x_0} [f(x) + g(x)] = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x)$

证明: 设 $\lim_{x \rightarrow x_0} f(x) = A$, $\lim_{x \rightarrow x_0} g(x) = B$

由 $\lim_{x \rightarrow x_0} f(x) = A \Rightarrow \forall \varepsilon > 0, \exists \delta_1 > 0$, 当 $x \in \dot{U}(x_0, \delta_1)$ 时 $|f(x) - A| < \frac{\varepsilon}{2}$

同理 对于相同的 $\varepsilon, \exists \delta_2 > 0$, 当 $x \in \dot{U}(x_0, \delta_2)$ 时, $|g(x) - B| < \frac{\varepsilon}{2}$

当 $\delta = \min(\delta_1, \delta_2)$, $x \in \dot{U}(x_0, \delta)$ 时, $|f(x) + g(x) - A - B|$

$$< |f(x) - A| + |g(x) - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\therefore \lim_{x \rightarrow x_0} [f(x) + g(x)] = \lim_{x \rightarrow x_0} f(x) + \lim_{x \rightarrow x_0} g(x)$$

3. 设 $f(x) = \frac{|x|}{x}$, 证明极限 $\lim_{x \rightarrow 0} f(x)$ 不存在

$$\text{证明: } \because \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = 1 \neq \lim_{x \rightarrow 0^-} f(x)$$

$\therefore$ 极限 $\lim_{x \rightarrow 0} f(x)$ 不存在.